

On the importance of astronomical refraction for modern Solar astrometric measurements

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Received March 31, 2013; accepted ??

ABSTRACT

Context. Several efforts are currently made from space missions in order to get accurate solar astrometric measurements i.e. to probe the long term variations of solar radius or shape, their link with solar irradiance variations and their influence on earth climate. These space missions use full disk solar imagery. In order to test our ability to perform such measurements from ground on the long term, we need to use similar techniques and instruments simultaneously from ground and space. This should help us to model and understand how the atmosphere affect ground based metrologic measurements. However, using full imagery from ground instead of the traditional astrolabe technique immediately raise the question of the effect of refraction and how well we can correct from it.

Aims. The goal is to study in details the influence of pure astronomical refraction on solar metrologic measurements made from ground-based full disk imagery and to provide the tools for correcting the measurements and estimating the associated uncertainties.

Methods. We use both analytical and numerical methods in order to confront commonly or historically used approximations and exact solutions.

Results. We provide the exact formulae for correcting solar radius measurements at any heliographic angle and for any zenith distance. We show that these corrections can be applied up to 80° of zenith distance provided that full numerical integration of the refraction integral is used. We also provide estimates of the absolute uncertainties associated with the differential refraction corrections and shows that approximate formulae can be used up to 80° of zenith distance for computing these uncertainties. For a given instrumental setup and the knowledge of the uncertainties associated with local weather records, this can be used to fix the maximum zenith distance one can observe depending on the required astrometric accuracy.

Key words. Atmospheric effects – Sun: fundamental parameters – Astrometry

1. Introduction

Ground based solar astrometric measurements have up to now been based on the so-called equal altitudes method (Débarbat & Guinot 1970). They have historically been made from transit instruments or astrolabes. Several instruments, derived from Danjon astrolabe, have been dedicated to solar diameter measurements, as DORaySol experiment (Morand et al. 2011). Observations consist in determining the transit times, through the same equal zenith distance circle, of the two solar limbs which are the extremities of a vertical solar diameter. Accuracy of these measurements is then mainly determined by datation accuracy and not by the optical resolution of the instrument. As the two limbs are observed at equal zenith distances, influence of astronomical refraction is inherently reduced (e.g. Laclare et al. 1996). Only the small climatic conditions variations (temperature, pressure, relative humidity) between the two crossings, distant from a few minutes of time, can still play a role. The main drawback of this method is that only vertical diameters can be determined.

Recent work in the field of solar metrology involve measurements from space using full disk solar images (Dame et al. 1999; Kuhn et al. 2012). PICARD-SOL (Meftah et al. 2013) is a ground based project that was set up at Calern observatory in

order to use the same technique simultaneously from ground and space and in order to inter-calibrate the different measurements. Using full disk imagery from ground raise however the question of the influence of astronomical refraction and how well we can correct for it.

The effect of atmospheric refraction is to change the true topocentric zenith angle z^t of a celestial object to a lower observed one z . The refraction function $R(z)$ is defined by:

$$z = z^t - R(z) \quad (1)$$

Alternatively, we may take the true angles as argument and define the associated refraction function \bar{R} by:

$$z = z^t - \bar{R}(z^t) \quad (2)$$

If the refraction function $R(z)$ is known, the associated function $\bar{R}(z^t)$ can easily be evaluated for any true zenith distance z^t by solving the non linear equation $x - R(z^t - x) = 0$.

2. fundamental equations for astronomical refraction

From Snell's law of refraction applied to a spherical atmosphere, the curvature of a light path is linked to the local refractive index n through the so-called refractive invariant :

$$n r \sin(\xi) = \text{constant} \quad (3)$$

where ξ is the local zenith distance i.e. the angle between the light ray and the radius vector r from Earth center. From this, the differential refraction along the light ray is obtained by:

$$dR = -\tan \xi \frac{dn}{n} \quad (4)$$

In order to find the total amount of refraction at observer position, we can integrate along the full ray path from $n = n_{\text{obs}}$ and $\xi = z$ at observer position up to $n = 1$ outside the atmosphere.

$$R = \int_1^{n_{\text{obs}}} \tan \xi \frac{dn}{n} \quad (5)$$

This can be done either by direct numerical integration of Eq. (5) after an appropriate change of variable (Auer & Standish 2000) or by using a full ray-tracing procedure solving the system of coupled differential equations provided by Eq. (4) and the differentiation of Eq. (3) (van der Werf 2003, 2008). This, in principle, requires a model of the full atmosphere i.e. temperature, pressure, density etc...at any point through the light path. In the next section we recall why this is in fact not needed if we avoid areas close to the horizon and give some usual approximations of the refraction integral.

3. Approximation to the refraction integral

For zenith distance up to 70° , the refraction integral can be evaluated with good accuracy without any hypothesis about the structure of the atmosphere: it depends only on temperature and pressure at the observer (Oriani's theorem, see also: Ball 1908; Young 2004). This justifies that, over time, a large number of nearly equivalent approximate formulae have been derived that do not require the full knowledge of the structure of the real atmosphere. A development of the refraction integral into semi-convergent series of odd power of $\tan(z)$ is what is commonly found in textbooks (e.g. Ball 1908; Smart 1965; Woolard & Clemence 1966; Danjon 1980). An example of this will be given in Sect. 3.1. In fact the first two terms of such expansion (up to \tan^3) corresponds to what is known as Laplace formulae of which Fletcher (1931) said that *no reasonable theory differs by more than a few thousandths, hundredths, tenths of a second at $z = 60^\circ, 70^\circ, 75^\circ$ respectively.*

For large zenith distance, $\tan(z)$ power series will diverge at the horizon and are not appropriate. Closed formula valid at low zenith distance and that are finite at the horizon can however still be found (see e.g. Wittmann 1997). Assuming an exponential law for the variation of air density with height, it's possible for instance to derive a formula involving the error function (Fletcher 1931; Danjon 1980). Another example is Cassini's exact formula for an homogeneous atmosphere model. While physically un-realistic, the model of Cassini, thanks to Oriani's theorem, gives also excellent results up to at least 70° of zenith distance while remaining finite down to the horizon (Young 2004). For large zenith distances however, Young (2004) have shown that the lowest layers of the atmosphere and especially the lapse rate at observer becomes progressively dominant as one observe

closer to the horizon. This therefore should be included in atmospheric models and we can not avoid anymore the full numerical evaluation of the refraction integral.

In the following sub-sections we present first in details the refraction model as it was used for reducing solar astrolabe data at Calern observatory, then we give the full error function model from which the Calern model was actually derived and finally we recall Cassini's formula. In Sect. 5, these three approximations will then be compared to full numerical integration of the refraction integral using a standard atmosphere model.

3.1. Refraction model used at Calern observatory for Solar metrology

The refraction model that was used for the reduction of astrolabe measurements at Calern observatory is a truncation of the expansion in odd power of $\tan(z)$ (Danjon 1980). For an observer at geodetic latitude φ and altitude h above the reference ellipsoid, the refraction R is obtained as a function of the observed zenith angle, the wavelength (λ) and local atmospheric conditions i.e. pressure (P), absolute temperature (T), and relative humidity ($f_h \in [0, 1]$) by:

$$R(z, \lambda, P, T, f_h, h, \varphi) = \alpha(1 - \beta) \tan(z) - \alpha\left(\beta - \frac{\alpha}{2}\right) \tan^3(z) + 3\alpha\left(\beta - \frac{\alpha}{2}\right)^2 \tan^5(z) \quad (6)$$

where

$$\alpha(T, P, f_h, \lambda) = n_{\text{obs}} - 1 \quad (7)$$

is the air refractivity for local atmospheric conditions and the given wavelength, and

$$\beta(T, h, \varphi) = \ell(T)/r_c(\varphi, h) \quad (8)$$

is the ratio between the height ℓ of the homogeneous atmosphere and the earth radius of curvature r_c at observer position. The homogeneous atmosphere has by definition a constant air density ρ equal to the one at observer position and its height is such that it would give the same pressure as the one recorded at observer position. Note that we do not assume here that the atmosphere is homogeneous, we just use the reduced height that can be obtained for any real atmosphere just from the pressure and density at observer. Assuming furthermore ideal gas law for dry air we have:

$$\ell(T) = \frac{P}{\rho g} = \frac{P_0}{\rho_0 g_0} \frac{T}{T_0}, \quad (9)$$

where $\rho_0 = 1.293 \text{ kg m}^{-3}$ for $T_0 = 273.15 \text{ K}$, $P_0 = 101325 \text{ Pa}$ and normal gravity $g_0 = 9.80665 \text{ m s}^{-2}$. The radius of curvature for Calern observatory ($\varphi = 43^\circ 45' 7''$, $h = 1323 \text{ m}$) was approximated by the minimum reference ellipsoid curvature at latitude 45° and sea level (Chollet 1981, see Appendix A):

$$r_c(45^\circ, 0) = 6367.512 \text{ km} \quad (10)$$

Ambient air refractivity was deduced from the refractive index $n_0(\lambda)$ under standard conditions and the partial pressure of water vapor p by applying the formula recommended by the first resolution of the 13th General Assembly of the International Union of Geodesy and Geophysics (IUGG 1963; Baldini 1963). After conversion to Pa (Pascal) as the pressure unit, the equation becomes:

$$\alpha(T, P, f_h, \lambda) = \frac{T_0}{T} \left\{ (n_0(\lambda) - 1) \frac{P}{P_0} - 4.13 \cdot 10^{-10} p(f_h, T) \right\} \quad (11)$$

Refractivity under standard condition (sea level, $T = T_0$, $P = P_0$, 0% humidity, 0.03% of carbon dioxide) was taken from the work of Barrel and Sears (1939):

$$n_0(\lambda) - 1 = \left\{ 2876.04 + \frac{16.288}{(10^6 \lambda)^2} + \frac{0.136}{(10^6 \lambda)^4} \right\} 10^{-7}. \quad (12)$$

Partial pressure of water vapor for the current temperature and relative humidity was deduced from a fit of water vapor pressure data published by the Bureau Des Longitudes (1975) for temperatures between -15°C and $+25^\circ\text{C}$. The resulting equation, converted to Pa, is (Chollet 1981):

$$p(f_h, T) = f_h 6.1075 10^2 e^{7.292 10^{-2}(T-T_0) - 2.84 10^{-4}(T-T_0)^2} \quad (13)$$

Finally, we note that local atmospheric pressure P was measured from the height H (in mm) of a mercurial barometer and its temperature θ (in $^\circ\text{C}$). Taking into account corrections for local gravity (latitude and altitude) and for temperature (through the volume thermal expansion of mercury and the coefficient of linear thermal expansion of the tube), P was obtained by¹ (see Appendix B):

$$P = H \left\{ 1 - 2.64 10^{-3} \cos(2\varphi) - 1.96 10^{-7} h - 1.63 10^{-4} \theta \right\} \quad (14)$$

3.2. Error function formula

In fact, in Eq. (6), only the first two terms which correspond to Laplace formula can be found without any hypothesis on the real atmosphere (only the reduced height ℓ and the refractivity at observer are needed). The term in \tan^5 comes from an additional assumption, namely the fact that air density follows an exponential decrease with height (actually with a well chosen variable which vary almost linearly with height, see Danjon (1980), Fletcher (1931)). This leads to the following equation:

$$R = \alpha \left(\frac{2 - \alpha}{\sqrt{2\beta - \alpha}} \right) \sin(z) \Psi \left(\frac{\cos(z)}{\sqrt{2\beta - \alpha}} \right) \quad (15)$$

with :

$$\Psi(x) = e^{x^2} \int_x^\infty e^{-t^2} dt = \frac{\sqrt{\pi}}{2} e^{x^2} (1 - \text{erf}(x)) \quad (16)$$

from which Eq. (6) was derived by keeping only the three first terms of its asymptotic expansion.

3.3. Cassini

By comparing the results with a full integration method, Young (2004) shows the superiority of Cassini's formula over the series-expansion approach and advocates its use by astronomers. Cassini assumed an homogeneous atmosphere for which he obtained the exact formula:

$$R = \text{asin} \left(\frac{n_{\text{obs}} r_c \sin(z)}{r_c + \ell} \right) - \text{asin} \left(\frac{r_c \sin(z)}{r_c + \ell} \right) \quad (17)$$

Again, it can be shown (Ball 1908) that expanding this formula also leads to the first two terms of Eq. (6) i.e. to Laplace formula.

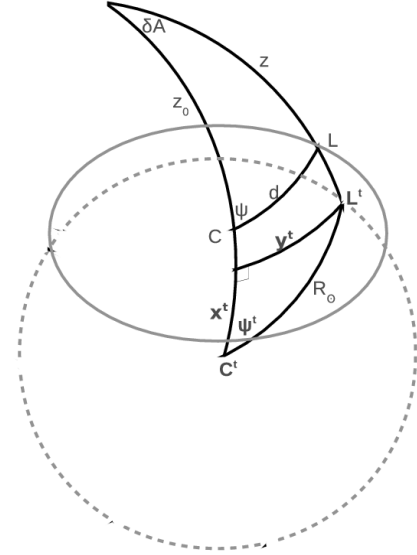


Fig. 1. Geometry for the solar shape due to astronomical refraction. The dashed circle represents the true solar disk of centre C' and radius R_\odot while the elliptical shape (full line) represents the observed Sun of centre C . The point at the top represents observer's zenith.

4. On the observed shape of the Sun due to pure astronomical refraction

In this section, we assume that the Sun is a perfect sphere of angular radius R_\odot at 1 AU and that there is no other effect affecting its observed shape than astronomical refraction defined by Eq. (2).

In the horizontal coordinate system (zenith distance-azimuth), we note (z_\odot, A_\odot) the true position of the Sun centre (C') observed at zenith angle z_\odot ; (z', A) the true position of a point (L') of the solar limb observed at zenith angle z ; $\delta z = z - z_\odot$ and $\delta A = A - A_\odot$. Figure 1 shows all the angles involved. Each true limb point position can be defined by the angle $\psi' \in [-\pi, \pi[$ between the direction $C'L'$ and the vertical circle. Similarly, each observed limb point can be located by the angle $\psi \in [-\pi, \pi[$ between the observed direction CL and the vertical circle. However, because the figure is symmetric with respect to the vertical circle, we consider only the interval $[0, \pi]$ for ψ and ψ' in the following. For observation with an Alt-Az mount this would correspond directly to the angle with one of the CCD axis. For an equatorial mount, one CCD axis is aligned with the hour circle passing through the celestial poles and the Sun and therefore the vertical circle can be materialized on the solar image by computing first the parallactic angle between these two circles.

If $d(\psi) = \bar{d}(\psi')$ is the angular distance between the observed position of the Sun centre and the observed limb points, we define by:

$$\langle d \rangle = \left[\frac{1}{\pi} \int_0^\pi d(\psi)^2 d\psi \right]^{1/2} = \left[\frac{1}{\pi} \int_0^\pi \bar{d}(\psi')^2 d\psi' \right]^{1/2} \quad (18)$$

the geometric mean radius of the observed Sun. The horizontal and vertical angular extent of the observed Sun are noted D_h and D_v respectively and, the flattening is given by:

$$f = \frac{D_h - D_v}{D_h} \quad (19)$$

¹ Chollet (1981) used erroneously $2.64 10^{-4}$ in this equation.

Following Mignard (2002), we define the magnification Γ as the ratio between the vertical size of the image (δz) of a small object to its true size ($\delta z'$). From Eqs. (1) and (2), we have:

$$\Gamma = \frac{dz}{dz'} = 1 - \frac{d\bar{R}}{dz'} = \left(\frac{dz'}{dz} \right)^{-1} = \left(1 + \frac{dR}{dz} \right)^{-1} \quad (20)$$

The distortion Δ is then defined as the rate of change of the magnification:

$$\Delta = \frac{d\Gamma}{dz} = -\Gamma^2 \frac{d^2 R}{dz^2} = -\frac{1}{\Gamma} \frac{d^2 \bar{R}}{dz'^2} \quad (21)$$

4.1. Approximate formulae for all zenith angles

Any limb point true position can be located by its projections on the vertical circle passing through the true Sun centre, and on the great circle perpendicular to this vertical circle passing through the limb point (see Fig. 1). Because all the angles involved are small, we can write:

$$x' = R_{\odot} \cos(\psi') \quad (22)$$

$$y' = R_{\odot} \sin(\psi') \quad (23)$$

and:

$$x'^2 + y'^2 = R_{\odot}^2 \quad (24)$$

By looking at the expression of the observed values x and y of these projections, one can obtain an approximate formula for the observed shape of the Sun.

The projection x' on the vertical circle can be approximated by keeping the two first terms of a Taylor expansion of the refraction:

$$x' \simeq z' - z'_{\odot} = \delta z + R(z) - R(z_{\odot}) \simeq \delta z \left(1 + \frac{dR}{dz} \right) + \frac{(\delta z)^2}{2} \frac{d^2 R}{dz^2} \quad (25)$$

The observed projection y is linked to z and δA both by the cosine and sine rules:

$$\cos(y') \simeq \cos^2(z') + \sin^2(z') \cos(\delta A) \quad (26)$$

$$\sin(y') = \sin(\delta A) \sin(z') \quad (27)$$

Differentiating Eq. (26) and using Eq. (27) with $\sin(y') \simeq y'$, $\sin(\delta A) \simeq \delta A$ and $dz' = -\bar{R}(z')$ leads to:

$$dy' = \frac{-y' \bar{R}(z')}{\tan(z')} \quad (28)$$

The observed distance y is then obtained by:

$$y \simeq \delta A \sin(z) = y' + dy' = y' \left(1 - \frac{\bar{R}(z')}{\tan(z')} \right) \quad (29)$$

Finally, by reporting Eqs. (25) and (29) in Eq. (24) and using Eqs. (20) and (21), we obtain:

$$\left[\frac{\delta z}{\Gamma} - \frac{\Delta}{2} \left(\frac{\delta z}{\Gamma} \right)^2 \right]^2 + \left[\frac{\delta A \sin z}{1 - \frac{\bar{R}(z')}{\tan(z')}} \right]^2 = R_{\odot}^2 \quad (30)$$

where the magnification and distortion are taken at z_{\odot} . From this we can deduce the position of the two vertical limb points

and the observed vertical extent of the image. For $\Delta \ll R_{\odot}$ and $\delta A = 0$, we find:

$$d(\pi) \simeq \Gamma R_{\odot} \left(1 + \frac{\Delta R_{\odot}}{2} \right) \quad (31)$$

$$d(0) \simeq \Gamma R_{\odot} \left(1 - \frac{\Delta R_{\odot}}{2} \right) \quad (32)$$

and thus:

$$D_v = d(0) + d(\pi) \simeq 2 \Gamma R_{\odot} \quad (33)$$

In the horizontal direction we obtain from Eq. (30) with $\delta z = 0$:

$$D_h = 2d(\pi/2) \simeq 2R_{\odot} \left(1 - \frac{\bar{R}(z'_{\odot})}{\tan(z'_{\odot})} \right) \quad (34)$$

4.2. Approximate formulae for small zenith angles - elliptic shape

Keeping only the first term in Eq. (6) is equivalent to neglecting Earth curvature. We obtain the following approximation valid close to the zenith only ($z < 45^\circ$):

$$R(z) = k \tan(z) \quad \text{with : } k = \alpha(1 - \beta) \quad (35)$$

For this flat-Earth approximation we can also write:

$$\bar{R}(z') \simeq k' \tan(z') \quad \text{with : } k' = k(1 - k \sec^2(z')) \quad (36)$$

In that case and if we neglect the distortion, Eq. (30) is reduced to the equation of a simple ellipse (see also e.g. Ball 1908):

$$\frac{x^2}{(1 - k' \sec^2(z'_{\odot}))^2} + \frac{y^2}{(1 - k')^2} = R_{\odot}^2 \quad (37)$$

where $x = \delta z$ and $y = \sin(z) \delta A$ can be assimilated to Cartesian coordinates on two perpendicular axes on the image. The major axis of the observed ellipse is thus given by:

$$\frac{D_h}{2} = R_{\odot}(1 - k') \quad (38)$$

while the observed minor axis is:

$$\frac{D_v}{2} = R_{\odot} (1 - k' \sec^2(z'_{\odot})) \quad (39)$$

We note from these equations that the Sun is shrunken in all directions. The observed horizontal diameter is smaller than the true diameter but remains the same for all zenith angles (c.f. Fig.7) while the observed vertical diameter decreases with increasing zenith distance. The combination of these two effects leads to the apparent flattening of the setting Sun (but keeping in mind that this approximate formula is not valid close to the horizon). From Eqs. (19), (38) and (39), the flattening for small zenith angles is:

$$f \simeq k \tan^2(z'_{\odot}). \quad (40)$$

while, near the horizon, Eq. (30) implies that the flattening is simply given by the vertical magnification taken at the the Sun's centre. For small zenith angles, the observed elliptic shape can be written as:

$$d(\psi) = \frac{D_v}{2 \sqrt{1 - (2f - f^2) \sin^2(\psi)}} \quad (41)$$

which can be approximated by:

$$d(\psi) \simeq R_{\odot} \left(1 - k' \left(1 + \cos^2(\psi) \tan^2(z'_{\odot}) \right) \right), \quad (42)$$

and the mean radius is obtained by:

$$\langle d \rangle = \frac{\sqrt{D_v D_h}}{2} \simeq R_{\odot} \left(1 - k' - \frac{k'}{2} \tan^2(z'_{\odot}) \right) \quad (43)$$

4.3. Exact formulae for all zenith angles

The classical approximate formulae above are useful for understanding the shape of the observed Sun in terms of magnification and distortion induced by refraction. Equation (30) shows that the general shape is a distorted ellipse with more flattening in the lower part than in the upper's. However, the shape of the observed Sun can also easily be obtained, in the general case, without any approximation. In the following, we obtain first the solution of the forward problem: for given true Sun radius R_{\odot} and true zenith distance z'_{\odot} , we obtain the shape of the observed Sun for any given refraction model. Then, we give the solution of the inverse problem: from the observed solar shape, the knowledge of z'_{\odot} (from ephemeris) and assuming a refraction model, we deduce the true angular solar radius.

4.4. Forward problem

Here we assume that the true zenith distance of the Sun centre z'_{\odot} and its true angular radius R_{\odot} are known. For any refraction model $\bar{R}(z')$, and true angle ψ^t , we deduce the observed angle ψ and angular distance $d(\psi)$. Applying the cosine and sine formulae respectively, we have :

$$\begin{cases} z^t = \arccos \left[\cos(z'_{\odot}) \cos(R_{\odot}) + \sin(z'_{\odot}) \sin(R_{\odot}) \cos(\psi^t) \right] \\ \delta A = \arcsin \left(\frac{\sin(R_{\odot}) \sin(\psi^t)}{\sin(z^t)} \right) \end{cases} \quad (44)$$

From Eq. (2), we can get the observed zenith distances:

$$z = z^t - \bar{R}(z^t) \quad \text{and} \quad z_{\odot} = z'_{\odot} - \bar{R}(z'_{\odot}) \quad (45)$$

and finally angular distances $\bar{d}(\psi^t)$ between the observed Sun centre and the observed positions of each limb point are obtained by application of the cosine rule:

$$\bar{d}(\psi^t) = d(\psi) = \arccos \left(\cos(z) \cos(z_{\odot}) + \sin(z) \sin(z_{\odot}) \cos(\delta A) \right) \quad (46)$$

where the observed angle ψ can be deduced from the true one by applying the sine rule:

$$\psi = \arcsin \left(\frac{\sin(\delta A) \sin(z)}{\sin(\bar{d}(\psi^t))} \right) = \arcsin \left(\frac{\sin(z)}{\sin(z^t)} \frac{\sin(R_{\odot})}{\sin(\bar{d}(\psi^t))} \sin(\psi^t) \right) \quad (47)$$

The smallest observed diameter of the Sun is obtained on the vertical direction:

$$D_v = d(0) + d(\pi) = 2R_{\odot} - \left(\bar{R}(z'_{\odot} + R_{\odot}) - \bar{R}(z'_{\odot} - R_{\odot}) \right) \quad (48)$$

and the largest angular extent, observed in the direction parallel to the astronomical horizon is obtained by:

$$D_h = 2d(\pi/2) \quad (49)$$

We note that Eqs. (44) and (45) lead back to the approximation Eq. (34) for the largest observed angular extent. This is however

more easily obtained using the sine rule rather than Eq. (46). With $\sin(d(\pi/2)) \simeq d(\pi/2)$, $\sin(R_{\odot}) \simeq R_{\odot}$ and $\cos(R_{\odot}) \simeq 1$, we obtain:

$$d(\pi/2) \simeq \sin(z) \sin(\delta A) = \sin(z'_{\odot} - \bar{R}(z'_{\odot})) \frac{R_{\odot}}{\sin(z'_{\odot})}. \quad (50)$$

which, with a first order expansion of the sine function around z'_{\odot} , leads to Eq. (34).

4.5. Inverse problem

Here we give the solution of the inverse problem: given a refraction model $(R(z), \bar{R}(z'))$, knowing z'_{\odot} from ephemeris and the observed angular distance $d(\psi)$ between the observed Sun centre and a limb point at an observed angle ψ with the vertical circle, we deduce the true angular radius R_{\odot} . One can compute successively:

$$\begin{cases} z_{\odot} = z'_{\odot} - \bar{R}(z'_{\odot}) \\ \delta A = \arctan \left[\frac{\sin(\psi)}{\sin(z_{\odot}) \cot(d(\psi)) + \cos(z_{\odot}) \cos(\psi)} \right] \\ z = \arcsin \left[\frac{\sin(\psi) \sin(d(\psi))}{\sin(\delta A)} \right] \\ z^t = z + R(z) \\ \psi^t = \arctan \left[\frac{\sin(\delta A)}{\cos(z'_{\odot}) \cos(\delta A) - \sin(z'_{\odot}) \cot(z^t)} \right] \\ R_{\odot} = \arcsin \left[\frac{\sin(\delta A) \sin(z^t)}{\sin(\psi^t)} \right] \end{cases} \quad (51)$$

5. Results

5.1. On the absolute value of refraction

We first look at the absolute value of refraction and compare the various approximate formulae of Sect. 3 to the full numerical integration of the refraction integral using a standard atmosphere (Sinclair 1982). This atmosphere is assumed to be spherically symmetric, in hydrostatic equilibrium and made of a mixture of dry air and water vapor that follows the perfect gas law. It is made of two layers: the troposphere with a constant temperature gradient which extends from the ground up to the tropopause at 11 km, and an upper isothermal stratosphere. Like in the US Standard Atmosphere (1976), the temperature and pressure at the surface are 288.15 K and 101325 Pa and the constant tropospheric lapse rate is 6.5 K km⁻¹. In the troposphere, the relative humidity f_h is assumed constant and equal to its value at the observer. The partial pressure of water vapor in a tropospheric layer at temperature T is then obtained by:

$$p(f_h, T) = f_h \left(\frac{T}{247.1} \right)^{\delta} 10^2 \quad (52)$$

which, with $\delta = 18.36$, never depart by more than 0.5 hPa from Eq. (13) for temperature lower than 30°. Dry air is assumed in the stratosphere. Finally, Eq. (11) and its derivatives with respect to T and P are used to find air refractivity along the integral path.

The numerical integration was performed by using the method of Auer & Sandish (2000) also recommended by the Astronomical Almanac (Seidelmann 1992). The program used

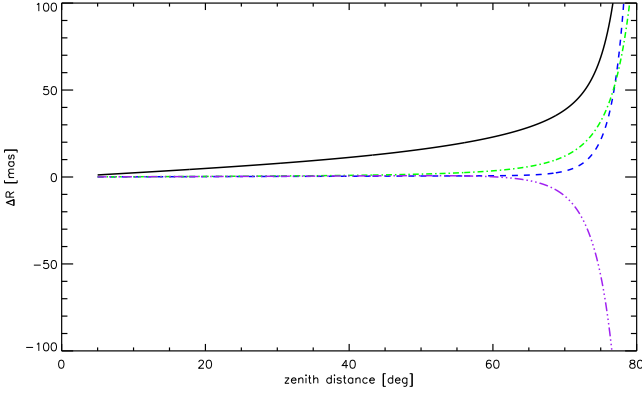


Fig. 2. Absolute differences (in mas) between a reference model and the different approximate refraction formulae as a function of the true zenith distance. The reference model is obtained by full numerical integration of the US Standard Atmosphere (1976) and Ciddor (1996) equation for air refractivity. From top to bottom: \tan^5 expansion Eqs. (6)-(13), full error function Eq. (15), \tan^5 expansion Eq. (6), Cassini's formula Eq. (17). All approximate formulae but the top one use Ciddor (1996) air refractivity

is based on the one published by Hohenkerk & Sinclair (1985) but adapted in order to use a dispersion equation based on the work of Peck & Reeder (1972) in replacement of the less accurate equation of Barrel & Sears (1939) (Eq. (12)). For $T = 15^\circ\text{C}$, $P = P_0$, 0% humidity and 0.045% of carbon dioxide, we take:

$$n_0(\lambda) - 1 = \left\{ \frac{0.05792105}{238.0185 - (10^6\lambda)^{-2}} + \frac{0.00167917}{57.362 - (10^6\lambda)^{-2}} \right\} \quad (53)$$

This dispersion equation was also used by Ciddor (1996) who derived a new set of equations for calculating the refractive index of air which was subsequently adopted by the International Association of Geodesy (IAG 1999) as a new standard. In the following, all computations have been made using $\lambda = 535.7\text{ nm}$ which is one of the wavelengths used by the PICARD-SOL project.

Figure 2 shows the absolute differences in milliarcseconds (mas) between the approximate formulae and the exact integral evaluation for zenith distances up to 80° . We immediately see that for zenith distance lower than 75° , all the approximate formulae lead to less than 50 mas of absolute error. The full line corresponds to the \tan^5 formula Eq. (6) described in Sect. 3.1 while the dashed line corresponds to the same formula but using the new Ciddor (1996) equations instead of Eqs. (11)-(13) for computing air refractivity. For zenith distances lower than 80° , the impact of using the old formula for refractivity never exceed 80 mas. The superiority of Ciddor equations to better fit observations and this for a wider range in wavelengths is however clearly established. The two other lines correspond to the error function (dot-dash) and Cassini (triple dots-dash) formulae both using the Ciddor (1996) equation for refractivity. These two last formulae were selected mainly because, unlike the series expansions in $\tan(z)$, they are finite at the horizon. The full integration with standard atmosphere conditions leads to a refraction of about $1980''$ at the horizon. The error function and Cassini formulae lead respectively to $2088''$ and $1180''$ corresponding to relative errors of 5% and 40% respectively. This tends to favour the use of the error function formula over Cassini's one very close to the

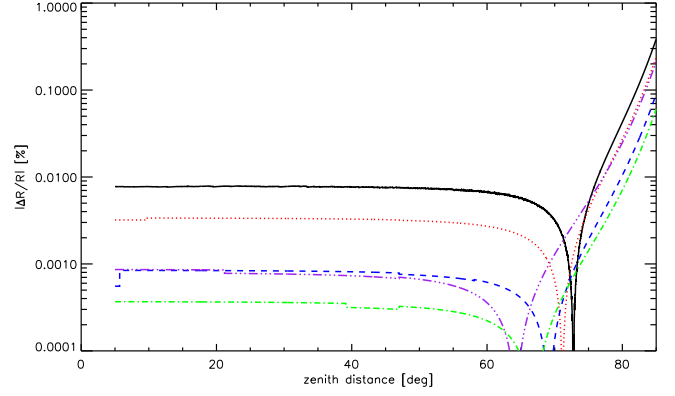


Fig. 3. Relative error on refraction as a function of zenith distance for different tropospheric lapse rate. The reference model use US Standard Atmosphere (1976) with a lapse rate of 6.8 K km^{-1} . The top curve correspond to an isothermal model and other atmosphere models have lapse rate of 2.5, 5, 10 and 7.5 K km^{-1} (from top to bottom at low zenith distance). All models are computed using full numerical integration.

horizon. The hypothesis made to derive the error function formula are indeed more realistic than Cassini's hypothesis of an homogeneous atmosphere. It has however been shown that refraction below 5° of the horizon is variable and strongly depend on the local lapse rate and properties of the boundary layer above or below the observer's eye (e.g. Young 2004). Within few degrees from the horizon, refraction may be influenced by thermal inversion boundary layers, ducting or other phenomena leading to extreme refraction. In this range, the local lapse rate must be known and it is not expected that any formula using just the temperature and pressure at observer could give an accurate absolute refraction.

It is however probably more interesting to look in the range between 60° and 85° of zenith distance, which is more important to astronomers willing to push in that range the limits of their astrometric measurements using only temperature and pressure recorded at observer position. We first note from Fig. 2 that, between 60° and 77° , the \tan^5 expansion formula is actually giving slightly better absolute refraction values than the error function formula. If we now assume that temperature and pressure at observer position are perfectly known, the only remaining important unknown in the atmospheric model is the tropospheric lapse rate. We can however fix limits for a realistic lapse rate: it must lie between an isothermal model and a lapse rate of 10 K km^{-1} which would correspond to an adiabatic atmosphere (Young 2004). Figure 3 shows the absolute value of the relative error for such models with lapse rate ranging from 0 to 10 K km^{-1} when they are compared to the standard model with a lapse rate of 6.8 K km^{-1} . From this we can deduce that, no matter what is the real atmosphere, if the conditions at observer are known, the relative error on refraction is lower than 0.01% for zenith angles below 77° and lower than 0.4% for zenith angles between 77° and 85° .

5.2. On the mean solar radius correction

Figure 4 shows the difference between the true radius of the Sun and the mean radius of the observed Sun as defined by Eq. (18) as a function of the true zenith distance of the centre of the Sun. The exact formula Eq. (46) was used and we took standard con-

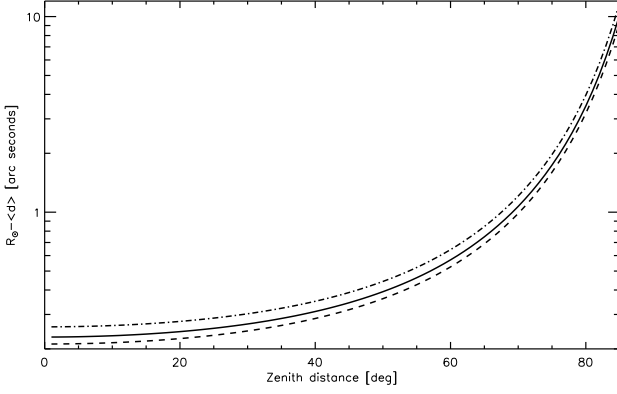


Fig. 4. Difference between the true solar radius and the observed one as a function of the true zenith distance. The full line corresponds to average weather conditions at Calern ($T = 15\text{ }^{\circ}\text{C}$, $P = 875\text{ hPa}$). The dashed and dot-dashed lines correspond respectively to $T = -10\text{ }^{\circ}\text{C}$, $P = 900\text{ hPa}$ and $T = 30\text{ }^{\circ}\text{C}$, $P = 850\text{ hPa}$. All calculations are made using the exact formulae Eqs. (18) and (46) for Calern station assuming 50% humidity.

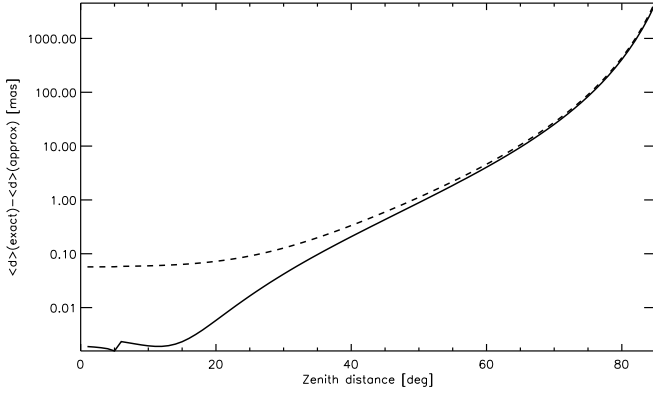


Fig. 5. Difference between the correction due to refraction on the mean solar radius as calculated from integrating the exact formula Eq. (46) or using the approximate formula Eq. (43). The dashed line is obtained by replacing k' by k in Eq. (43)

ditions for Calern observatory ($T = 15\text{ }^{\circ}\text{C}$, $P = 875\text{ hPa}$). The dashed and dot-dashed lines are for $T = -10\text{ }^{\circ}\text{C}$, $P = 900\text{ hPa}$ and $T = 30\text{ }^{\circ}\text{C}$, $P = 850\text{ hPa}$ respectively in order to illustrate the maximum amplitude of the effect at Calern station. The difference in the mean radius correction between the two extreme weather conditions range from 50 mas at the zenith up to 1850 mas at $z_{\odot}^t = 85^{\circ}$. It reaches 100 mas around $z_{\odot}^t = 55^{\circ}$ and 200 mas around $z_{\odot}^t = 70^{\circ}$. This represents always less than 0.2% of the correction.

Figure 5 shows the difference between the exact formula obtained by integrating Eq. (46) and the approximate formula Eq. (43) corresponding to an elliptical shape. The dashed line illustrates the result if k' is approximated by k (see Eq. (36)). In both cases the difference remains less than 20 mas for zenith distances lower than 70° . For larger zenith distances however, errors increase rapidly and the refraction function should be evaluated using full numerical integration.

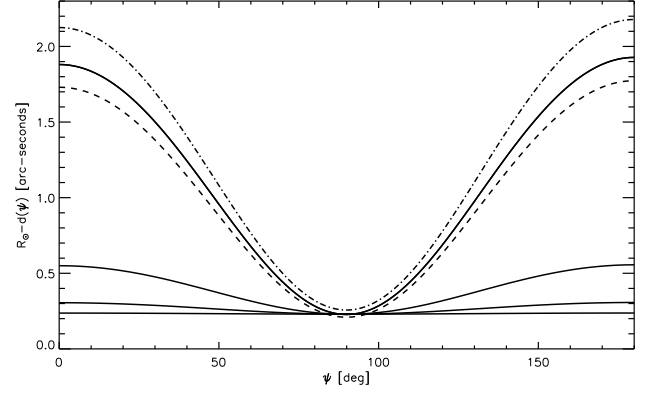


Fig. 6. Difference between the true solar radius and the angular distances between the observed Sun centre and the observed positions of each limb points between the vertical (north for $\psi = 0^{\circ}$ and south for $\psi = 180^{\circ}$) and the horizon ($\psi = 90^{\circ}$). The full lines are for $z_{\odot}^t = 70^{\circ}, 50^{\circ}, 30^{\circ}$ and 10° respectively from top to bottom and are for average weather conditions at Calern. The dashed and dot-dashed lines are for $z_{\odot}^t = 70^{\circ}$ and the same extreme weather conditions as in Fig. 4.

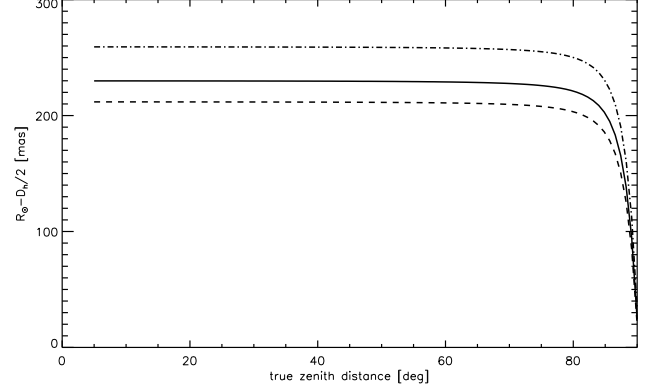


Fig. 7. Contraction of the horizontal radius ($R_{\odot} - d(\pi/2)$) as a function of the true zenith distance z_{\odot}^t . The full line is for average weather conditions at Calern. The dashed and dot-dashed lines are for the same extreme weather conditions as in Fig. 4.

5.3. On the angular dependence of solar radius correction

For precise metrologic measurements of the Sun and in order to correct for other effects (optical aberrations, turbulence, etc..) that are dependent on the position on the image, one may want to correct not the mean radius but each individual radius measured at all angles ψ . This can be done by following the procedure given in Sect. 4.5. Figure 6 is obtained from Eq. (46) and illustrates the amplitude of the correction as a function of ψ for different values of z_{\odot}^t , the true zenith distance of the Sun centre. We see that the horizontal diameter ($\psi = 90^{\circ}$) is affected by refraction (by about $2 \times 0.23'' = 0.46''$ for the chosen weather conditions) in agreement with Eq. (34). The north and south vertical corrections ($\psi = 0^{\circ}$ and 180° respectively) are also slightly different in agreement with Eqs. (31)-(32). Figure 7 shows that the contraction of the horizontal radius lies between 210 and 260 mas depending on the actual weather conditions and remains constant for all zenith distances below 80° . It then decreases rapidly towards zero at the horizon.

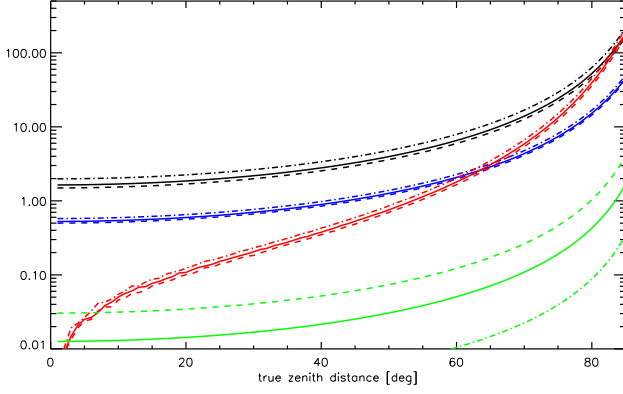


Fig. 8. Partial derivatives of the vertical diameter correction (δ_v) as a function of the true zenith distance. Partial derivatives in temperature, pressure, zenith distance and relative humidity are given in mas K^{-1} , mas hPa^{-1} , mas arcmin^{-1} and mas/\% from top to bottom (at 40°) respectively. The full, dashed and dot-dashed lines are for the same weather conditions as on Fig. 4.

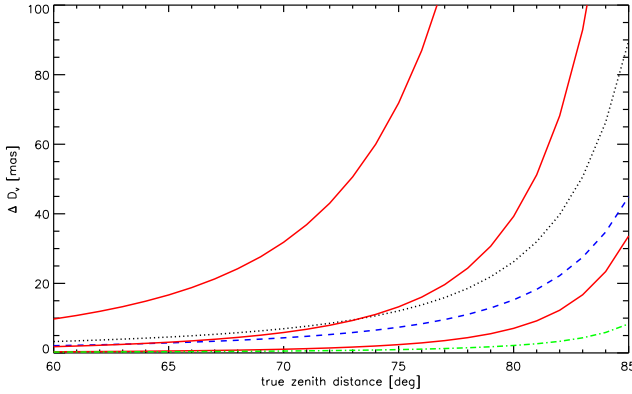


Fig. 9. Uncertainties on the vertical diameter correction assuming $\Delta T = 0.5 \text{ K}$ (dotted line), $\Delta P = 1 \text{ hPa}$ (dashed line), $\Delta f_h = 5\%$ and $\Delta z_\odot^t = 5.4', 1.0'$ or $0.2'$ (full lines from top to bottom). The total error is obtained by summing the four contributions.

5.4. On uncertainties associated to radius corrections

We have shown that, apart from weather conditions at observer's position, differences in atmospheric models and especially different tropospheric lapses rate will not play any significant role at least up to 85° of zenith distance. The four main contributions are therefore uncertainties in temperature, pressure, humidity and, for large zenith distance, uncertainties on the true zenith distance itself.

$$\Delta d(\psi) = \sum_{i=1}^4 \left| \frac{\partial d(\psi)}{\partial X_i} \right| \Delta X_i \quad X = \{T, P, f_h, z_\odot^t\} \quad (54)$$

It should be noted that we assume here observations made using filters with a narrow bandwidth around λ . For broadband filters, an additional term $\partial d(\psi)/\partial \lambda$ should be added by differentiating Eq. (53). The largest uncertainty will be obtained for the vertical diameter ($D_v = d(0) + d(\pi)$) which is the most affected by refraction. Figure 8 shows the four partial derivatives contributing to ΔD_v between the two extreme weather conditions chosen above for Calern (see Sect. 5.2). The partial derivatives shown

have been obtained by numerically differentiating Eq. (46) but we have also checked that the analytical expressions that can be derived from the approximate elliptical shape Eq. (42) are actually valid up to 80° of zenith distance. Closer to the horizon the partial derivative over the zenith distance becomes significantly overestimated (c.f. Fig. 10). From Eq. (42) and taking $k' \simeq k$, we obtain:

$$\begin{cases} \left| \frac{\partial d(\psi)}{\partial X_i} \right| = \left| \frac{\partial k}{\partial X_i} \right| (1 + \cos^2(\psi) \tan^2(z_\odot^t)) R_\odot & i = 1..3 \\ \left| \frac{\partial d(\psi)}{\partial z_\odot^t} \right| = 2k \cos^2(\psi) \sec^2(z_\odot^t) \tan(z_\odot^t) R_\odot \end{cases} \quad (55)$$

and from Eqs. (8), (9), (11), (36), (52), we obtain:

$$\begin{cases} \frac{\partial k}{\partial T} = -C_1 \frac{P}{T^2} (n_0(\lambda) - 1) + C_3 T^{\delta-1} f_h \left(C_2 - \frac{\delta-1}{T} \right) \\ \frac{\partial k}{\partial P} = C_1 (T^{-1} - C_2) (n_0(\lambda) - 1) \\ \frac{\partial k}{\partial f_h} = -C_3 (T^{-1} - C_2) T^\delta \end{cases} \quad (56)$$

where:

$$C_1 = T_0/P_0, \quad C_2^{-1} = C_1 r_c \rho_0 g_0, \quad C_3 = 4.13 \cdot 10^{-8} T_0 (247.1)^{-\delta} \quad (57)$$

For temperature, pressure and humidity, we assume uncertainties of $\Delta T = 0.5 \text{ K}$, $\Delta P = 1 \text{ hPa}$ and $\Delta f_h = 5\%$ which are typical for a standard weather station. The precision on the true zenith distance relies on ephemeris calculations and a correct timing. At any given time ephemeris can give not only z_\odot^t but also the instantaneous rate dz_\odot^t/dt and, from the knowledge of the image exposure time Δt , one can deduce an uncertainty on z_\odot^t by:

$$\Delta z_\odot^t = \left| \frac{dz_\odot^t}{dt} \right| \Delta t \quad (58)$$

The maximum rate is about $650'' \text{ min}^{-1}$ at summer solstice. Image exposures of 1 s, 5.5 s or 30 s would then correspond to a maximum uncertainty Δz_\odot^t of 0.18', 1' or 5.4' respectively. Figure 9 shows the contribution of these uncertainties to the total uncertainty on vertical diameter correction for large zenith distances. We can see for instance that for 1' precision on the zenith distance (or 5.5 s exposure), the uncertainty coming from zenith distance can become, above 70° , of the same importance as the combined uncertainties coming from temperature and pressure records. The total relative error on the vertical diameter correction ($\Delta D_v / (2R_\odot - D_v)$) remains however below 1% up to $z_\odot^t = 85^\circ$.

6. Conclusions

We have obtained in Sec. 4.5, the exact formulae that can be used to correct solar radius measurements at any heliographic angle and any zenith distance from the effect of astronomical refraction. Using full integration of the refraction integral, we have shown that this correction can be applied for any true zenith distance up to 85° with a relative uncertainty on the correction that remains less than 1%. Absolute uncertainties on these corrections are also derived that allows us to fix the maximum zenith distance one should observe depending on the needed metrologic accuracy. Figure 10 shows the maximum total absolute

uncertainty obtained on the solar radius assuming that the vertical radii have been observed at different zenith distances. We use $\lambda = 535.7$ nm and two exposure times used by the PICARD-SOL project at this wavelength. Because we took the maximum value for dz_{\odot}^t/dt , this curves represent only upper limits, the actual value of dz_{\odot}^t/dt should be use for each measurement. From this, one can deduce that observing below 70° , 75° or 80° of zenith distances will keep the absolute uncertainties on refraction corrections below 10, 20 and 50 mas respectively. The comparison between numerical derivatives (full lines) and the use of approximate formulae Eqs. (54)-(58) (dashed lines) shows that, even if the approximate formulae should not be used above 70° for correcting the measurements (c.f. Fig. 5), they can be used at least up to $z_{\odot}^t = 80^\circ$ for estimating the uncertainties.

In summary, the process that we suggest to correct ground based radii measurements from refraction for true zenith distances up to 80° is as follow. Inputs are: the measurements $d(\phi)$ and eventually their associated errors $\delta d(\phi)$ where ϕ is an arbitrary angle defined on the solar image; the time of image record and the exposure time Δt ; weather records (P , T , f_h) and their associated uncertainties (ΔT , ΔP and Δf_h); the wavelength (λ) and observer's geodetic coordinates (φ , h). One can then successively:

- find the direction of the zenith on the image and associate each angle ϕ to its corresponding angle ψ (c.f. Fig. 1). Depending on the instrumental setup, this may require the computation of the parallactic angle from ephemeris,
- determine z_{\odot}^t and dz_{\odot}^t/dt from ephemeris at the time of image record,
- calculate R_{\odot} using Eqs. (51) and full numerical integration for the refraction function $R(z, \lambda, P, T, f_h, h, \varphi)$,
- estimate $\Delta d(\psi)$ from Eqs. (54)-(58) and the knowledge of ΔT , ΔP , Δf_h , Δt and dz_{\odot}^t/dt ,
- estimate ΔR_{\odot} from:

$$\Delta R_{\odot} = R_{\odot} \frac{\Delta d(\psi) + \delta d(\psi)}{d(\psi)}. \quad (59)$$

For zenith distances lower than 70° full numerical integration can be replaced by Eq. (6) in order to evaluate the refraction function (c.f. Fig. 2). In both cases Ciddor (1996) equations should be used for computing air refractivity at observer position. The corresponding codes are available from the authors upon request.

It is important to keep in mind that, at all zenith distances, other phenomena such as extinction or optical turbulence must be taken into account for ground based solar metrology. We know that they will dominate refraction effects at low zenith distances. Close to the horizon extinction is proportional to refraction (Laplace's extinction theorem) and effects of optical turbulence (e.g. Ikhlef et al. 2012 and reference therein) will become increasingly important knowing that the Fried parameter varies as $\sec(z)^{-0.6}$. It is interesting however to know that for any zenith distance up to 80° refraction can be reliably corrected and uncertainties on this correction estimated. After these correction are applied, all other phenomena impacting metrologic measurements can therefore be investigated without fearing contamination by astronomical refraction even at high zenith distances. The mean radius correction presented here (c.f. Fig. 4) as well as mean turbulence corrections have been applied to correct the first PICARD-SOL measurements (Meftah et al. 2013). The corrections that can be applied individually for each heliographic

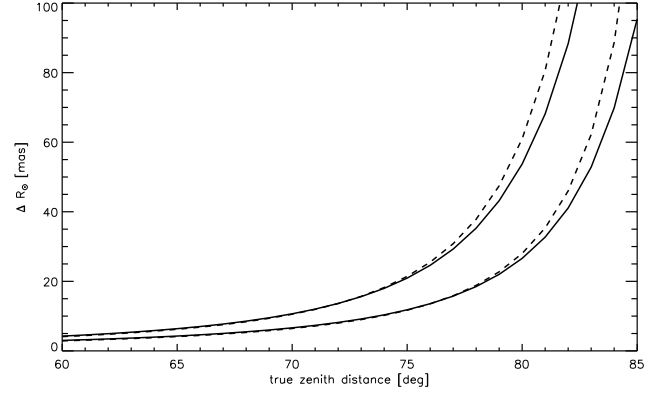


Fig. 10. Maximum absolute uncertainties on radius estimates using measurements in the vertical direction corrected from refraction. This curves are obtained for $\lambda = 535.7$ nm, $T = (15 \pm 0.5)^\circ\text{C}$, $P = (875 \pm 1)$ hPa, $f_h = 50\% \pm 5\%$ and $dz_{\odot}^t/dt = 650'' \text{ min}^{-1}$. The top and bottom lines are for 8.9 s and 1.3 s of exposure time respectively. Full lines give the results from full numerical derivatives calculations while dashed lines are obtained using approximate formulae Eqs. (54)-(58).

angles should be used in future work in order to disentangle the different effects.

Finally we note that we have considered only the radial symmetric-component of refraction also called pure or normal refraction. There also exists an asymmetric component known as anomalous refraction (e.g. Teleki 1979) resulting from the tilted atmospheric layers. Anomalous refraction may depend not only on zenith distance but also on azimuth and it can lead to seasonal or high frequency effects (see e.g. Hirt (2006) and references therein). The amplitude of such effect has however been found to be lower than $0.2''$ for local effects and one order of magnitude less for regional effects that may originate higher in the atmosphere (e.g. Hu 1991). Moreover it has been shown that anomalous refraction is spatially coherent at scales of at least 2° (Pier et al. 2003) and it has been established from dedicated observations that its main source is confined in the layer immediately above ground level (less than 60 m, see Taylor et al. (2013)). It is therefore difficult to believe that differential effects of anomalous refraction and especially the one that may be triggered in the Upper Troposphere - Lower Stratosphere (UTLS) interface (c.f. Badache-Damiani et al. 2007) could lead to significant bias on solar astrometric measurements relying on direct solar disk imaging.

Acknowledgements. The authors acknowledge financial support from CNES in the framework of PICARD-SOL project. We thank P. Exertier and J. Paris for useful discussions on the precise geodetic coordinates of Calern station, S. Y. van der Werf (Univ. of Groningen) for providing is ray tracing code also used to check our results and K. Reardon (INAF) for making available his IDL codes for computing refractivity from Ciddor (1996) equations.

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Appendix A: Note on the radius of curvature at Calern observatory

According to the WGS84 reference ellipsoid, the Earth's equatorial and polar radii are given respectively by $a = 6378.137$ km and $b = 6356.752$ km. The curvature in the (north-south) meridian and at the geodetic latitude of Calern solar astrometric instruments $\varphi = 43^\circ 45' 7''$ is then given by:

$$r_c^0 = \frac{(ab)^2}{(a^2 \cos^2(\varphi) + b^2 \sin^2(\varphi))^{3/2}} = 6365.985 \text{ km} \quad (\text{A.1})$$

One could also consider the mean radius of curvature calculated for Calern. From the curvature in the prime vertical (normal to the meridian):

$$r_c^{90} = \frac{a^2}{\sqrt{a^2 \cos^2(\varphi) + b^2 \sin^2(\varphi)}} = 6388.371 \text{ km} \quad (\text{A.2})$$

we can deduce the radius of curvature for any azimuth angle A by:

$$r_c^A = \frac{1}{\frac{\cos^2(A)}{r_c^0} + \frac{\sin^2(A)}{r_c^{90}}} \quad (\text{A.3})$$

from which we can deduce the mean radius of curvature averaging over all directions, by:

$$\langle r_c \rangle = \sqrt{r_c^0 r_c^{90}} = \frac{a^2 b}{a^2 \cos^2(\varphi) + b^2 \sin^2(\varphi)} = 6377.168 \text{ km} \quad (\text{A.4})$$

If, instead of the radius of curvature, one considers the distance from geocenter, we have:

$$R = \sqrt{\frac{a^4 \cos^2(\varphi) + b^4 \sin^2(\varphi)}{a^2 \cos^2(\varphi) + b^2 \sin^2(\varphi)}} = 6367.955 \text{ km} \quad (\text{A.5})$$

One should add to these values the elevation of the observer above the reference ellipsoid ($h = 1.323$ km for Calern observatory). If we consider that, on average, we observe the sun closer to the north-south direction than east-west direction we can take:

$$r_c = r_c^0 + h = 6367.308 \text{ km} \quad (\text{A.6})$$

which is very close to the value used by Chollet (1981).

Finally we note that, for ephemeris calculations, the geodetic latitude should be corrected for the local gravimetric deflection. For Calern solar astrometric instruments this lead to an astronomic latitude $\varphi_{\text{ast}} = 43^\circ 44' 53''$ which is also compatible within $1''$ with the direct measurements made using a full entry pupil astrolabe on the same site. Similarly, we note that taking into account the local undulation with respect to the reference ellipsoid leads to a height above sea level of $h_{\text{sl}} = 1.271$ km for Calern solar astrometric station.

Appendix B: Note on the corrections applied to mercurial barometer reading

The two corrections (for gravity and barometer temperature) can be written as multiplicative factors (e.g. Princo 2007):

$$P = H \left(\frac{1 + L \theta}{1 + M \theta} \right) \frac{g}{g_0} \quad (\text{B.1})$$

where P is the corrected atmospheric pressure, H is the barometer reading, $M = 1.818 \cdot 10^{-4} \text{ K}^{-1}$ is the coefficient of volume thermal expansion of mercury, and $L = 1.84 \cdot 10^{-5} \text{ K}^{-1}$ is the coefficient of linear thermal expansion of brass. According to the 1967 reference system formula (Helmert's equation), we have:

$$g = g_{45} (1 - a \cos(2\varphi) - b \cos^2(2\varphi)) \quad (\text{B.2})$$

where $g_{45} = 9.8061999 \text{ ms}^{-2}$ is the gravity acceleration at mid latitude, $a = 2.64 \cdot 10^{-3}$ and $b = 1.96 \cdot 10^{-6}$. This can be corrected from the so-called Free Air Correction (FAC) which accounts for the fact that gravity decreases with height above sea level ($C_{\text{FAC}} = -3.086 \cdot 10^{-6} \text{ s}^{-2}$), itself corrected in order to take into account the increasing gravity due to the extra mass assumed for a flat terrain (Bouger correction, $C_{\text{B}} = 4.2 \cdot 10^{-10} \text{ m}^3 \text{ s}^{-2} \text{ kg}^{-1}$). For a mean rock density of $\rho_r = 2.67 \cdot 10^3 \text{ kg m}^{-3}$ this leads to:

$$C_g = (C_{\text{FAC}} + \rho_r C_{\text{B}}) = -1.96 \cdot 10^{-6} \text{ s}^{-2} \quad (\text{B.3})$$

Close to 45° of latitude, the second term of Eq. (B.2) can be neglected and, if we note $\epsilon = 1 - g_{45}/g_0 = 4.6 \cdot 10^{-5}$, Eq. (B.1) can be approximated by:

$$P = H (1 - \epsilon) (1 - (M-L) \theta) \left\{ 1 - a \cos(2\varphi) + \frac{C_g}{g_{45}} h \right\} \quad (\text{B.4})$$

Neglecting second order terms leads to Eq. (14).

We note that absolute gravity measurements have now been made at Calern geodetic observatory leading to $g = (980215549.2 \pm 12.6) 10^{-8} \text{ m s}^{-2}$ (Nicolas et al. 2006). This shows that the relative error on the correction g/g_0 discussed above and previously used for refraction calculations was less than $5 \cdot 10^{-5}$. One could however now directly use Eq. (B.1) with the measured value of local gravity.